1st Annual Lexington Mathematical Tournament
Guts Round

Solutions

1. Answer: $\boxed{137/60}$

Solution: Putting the fractions over a common denominator, we get $\dfrac{60+30+20+15+12}{60} = \dfrac{137}{60}$.

2. Answer: $\boxed{50/3}$

Solution: Increasing a number by 20% is the same is multiplying by $1.2 = 6/5$. Thus, $6X/5 = Y$, and $5Y/6 = X$. This means we have to multiply Y by $5/6$ to get X, or decrease it by $1/6$, which, as a percent, is $100/6 = 50/3$.

3. Answer: $\boxed{4}$

Solution: If the radius is r, we have $2\pi r = 8\pi$, and solving gives $r = 4$.

4. Answer: $\boxed{4}$

Solution: Let the length of a side of the square be s. The perimeter is $4s$, and the area s^2, so $s^2 = 4s$. Since $s > 0$, we can divide both sides by s to get $s = 4$.

5. Answer: $\boxed{L}$

Solution: Observe that $4^3 = 64$. There are 26 letters, and Big Welk can make two cycles through them, writing down 52 letters. Then, the 64th letter he writes down will be the 12th letter of the alphabet, since he starts again from the 53rd letter. This gives the answer, L.

6. Answer: $\boxed{8}$

Solution: For the first part, Al travels 20 miles per hour for two hours, for a total of 40 miles. For the second part, Al travels 4 miles per hour for 6 hours, for a total of 24 miles. Thus, he travels 64 miles in 8 hours, so his speed is $64/8 = 8$ miles per hour.

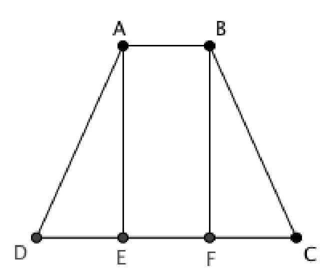
7. Answer: $\boxed{1/24}$

Solution: There are $4! = 24$ ways to re-assign the lunches to the students; the first student will receive one of the 4 lunches at random, the second will receive one of the 3 remaining ones at random, etc. However, only one of these re-assignments is the correct one, so our probability is $1/24$.

8. Answer: $\boxed{71}$

Solution: Notice that $111111 = 111 \cdot 1001 = 3 \cdot 37 \cdot 7 \cdot 11 \cdot 13$, so our answer is $3+37+7+11+13 = 71$ (It's a nice trick to remember that $abcabc$, with a, b, c digits, is equal to $1001 \cdot abc$, and furthermore that $1001 = 7 \cdot 11 \cdot 13$).

9. Answer: $\boxed{120}$



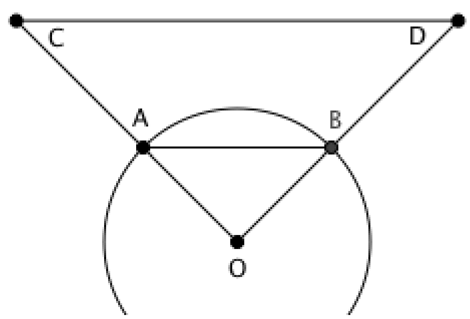
Solution: Let $ABCD$ be our trapezoid, with $AB = 5, CD = 15, BC = DA = 13$. Drop perpendiculars BF and AE to CD. Because the trapezoid is isosceles, we have $CF = ED$. Now, $ABFE$ is a rectangle, so $EF = 5$, and it follows that $DE = FC = 5$ because $CD = 15$. $AD = 13$, so by the Pythagorean Theorem, $AE = \sqrt{13^2 - 5^2} = 12$, meaning that the height of the trapezoid is 12. The average of the lengths of the bases is $(5 + 15)/2 = 10$, so our area is $10 \cdot 12 = 120$.

10. Answer: $\boxed{67}$
Solution: There are 3 possible pairs of digits that sum to 13: 4 and 9, 5 and 8, and 6 and 7. Our primes have to end in an odd number, so our only candidates are 49, 85, and 67. The first two are divisible by 7 and 5, respectively, and we find that 67 is prime, so this is our answer.

11. Answer: $\boxed{6}$
Solution: Each game has a winner and a loser, so the total number of wins of the four must equal the total number of losses. The total number of wins is $5+4+1+4 = 14$, and the total number of losses is $0 + 2 + 6 = 8$ without Ted's losses, so Ted must lose a total of 6 more games to equalize the number of wins and losses.

12. Answer: $\boxed{628}$
Solution: Rearranging our inequalities, we see that $a \leq e \leq b \leq d$. But d cannot be the largest of all five, since it is not equal to 5, so $c \geq d$. This gives $(a, e, b, d, c) = (1, 2, 3, 4, 5)$, and $a^b + c^d + e = 1^3 + 5^4 + 2 = 628$ (note that $c \geq a$ was unnecessary to solve the problem).

13. Answer: $\boxed{(15\sqrt{2} - 10)/2}$



Solution: We have $OA = OB = 5$ and $\angle OAB = 90°$. It follows that AOB is a 45-45-90 triangle. Since CD is parallel to AB, we have, for example, $\angle DCA = \angle BAO = 45°$, and we get that OCD is also a 45-45-90 triangle. CD is 200% more than AO, so it is 3 times as long, and we get $CD = 15$. Thus, $OC = \dfrac{15\sqrt{2}}{2}$, and $AC = OC - AO = \dfrac{15\sqrt{2}}{2} - 5 = \dfrac{15\sqrt{2} - 10}{2}$.

14. Answer: $\boxed{6}$
Solution: We only need to count the number of ways to position the green shirts, since the red shirts will have to go where the green shirts are not. If we let the positions to fill be A, B, C, D, and E, in that order, we want to choose two positions in which Seongcheol puts a green shirt; there are $\binom{5}{2} = 10$ ways to do this. However, in exactly four choices, A/B, B/C, C/D, and D/E, the two green shirts will be next to each other, so our answer will be $10 - 4 = 6$ (alternatively, we can simply list all ten possible pairs of positions for the green shirts and count the number of allowed orderings).

15. Answer: $\boxed{11}$
Solution: Expanding both sides using the distributive property, we get $x^2 + 4xy + 3y^2 = x^2 + 4xy + 4y^2$. We can subtract $x^2 + 4xy + 3y^2$ from both sides, to get $0 = y^2$, which means $y = 0$. Thus,

any solution $(x, 0)$ will work, and conversely, we need $y = 0$ to have a solution. There are a total of 11 possible values for x, corresponding to 11 solutions in total.

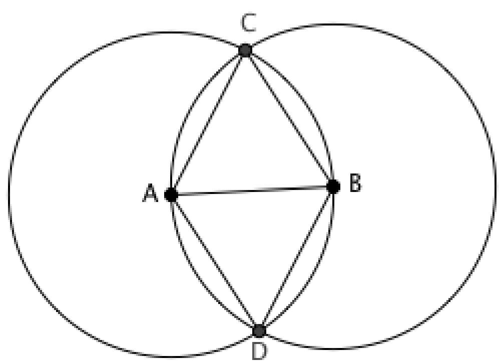
16. Answer: $\boxed{2/27}$

Solution: First, note that for Al to get two blue marbles, he cannot choose Bag 3, since it contains no blue marbles. Thus, the two bags he choose must be 1 and 2, which is equivalent to not choosing Bag 3. The probability of this happening is thus 1/3. Then, the probability of drawing a blue marble from Bag 1 is 2/3, and the probability of drawing a blue marble from Bag 2 is 1/3, giving a final probability of $(1/3)(2/3)(1/3) = 2/27$.

17. Answer: $\boxed{172}$

Solution: Noting that $90! = 90 \cdot 89!$, we can rewrite the sum as $89!(1 + 90) = 91 \cdot 89! = 7 \cdot 13 \cdot 89!$ by the distributive property. 89! is the product of the positive integers from 1 to 89, and thus, no prime greater than 89 can divide $89! + 90!$. We are now looking for the greatest prime numbers in the prime factorization of $7 \cdot 13 \cdot 89 \cdot 88 \cdots 1$, which are equivalently the largest primes less than equal to 89. 89 is prime, and the next largest prime is 83, giving us a sum of 172.

18. Answer: $\boxed{4\pi/3}$

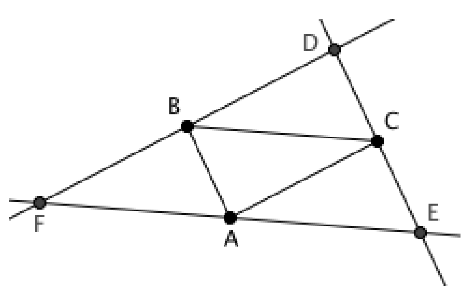


Solution: Let the centers of the circles be A and B, and let the circles intersect at C and D. All the shown segments, AB, AC, AD, BC, and BD, are radii of one of the circles, so their lengths are all equal to 1. Thus, triangles CAB and DAB are both equilateral. Thus, $\angle CAD = 2 \cdot 60° = 120°$, and arc CD of circle A is 1/3 the circumference of the circle. Similarly, arc CD of circle B is 1/3 the circumference of the circle. The two arcs together make up the perimeter of R, and are 2/3 the circumference of a unit circle, so our answer is $\frac{2}{3} \cdot 2\pi = \frac{4\pi}{3}$.

19. Answer: $\boxed{2}$

Solution: We have $f(x) = (x - 1)^2$ and $f(x + k) = (x + 1)^2$. Substituting $x \to x + k$ in the first equality, we have $f(x + k) = (x + k - 1)^2$, and we see that we want $x + k - 1 = x + 1$, so $k = 2$.

20. Answer: $\boxed{12}$



Solution: Let $A = (2, -4)$, $B = (-2, 8)$, and $C = (12, 7)$, and let the three points such that A, B, C, and the point form a parallelogram be D, E, F. Note that D, E, F can be obtained by intersecting the parallels to BC through A, to CA through B, and to AB through C. This is because, for example, if we let the parallels to AB and BC intersect at E, we have AE parallel to BC and AB parallel to EC, so we get a parallelogram $AECB$.

To find the coordinates of point E, note that translating the plane so that B goes to A also sends C to E. We can represent this translation by adding the same constants to the coordinates of all the points in the plane. That is, if we translate the plane so that B goes to A, we want to add 4 to each of the x-coordinates of the points in the plane, and subtract 12 from the y-coordinates, because, in some sense, $(-2, 8)+(4, -12) = (2, -4) = A$. Thus, to get E, we take $(12, 7)+(4, -12) = (16, -5)$.

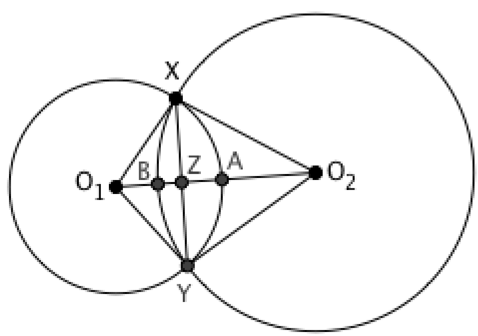
In a similar way, we find that $D = (8, 19)$, since the translation from C to D is the same as the translation from A to B, and that $F = (-12, -3)$, since the translation from C to A is the same as the one from B to F. Thus, our answer is $16 + 8 - 12 = 12$.

Comment: You may notice that the sum of the x-coordinates of D, E, F is the same as that of A, B, C. This isn't a coincidence! In fact, the same thing holds true of the y-coordinates. Try to figure out why this is true in general.

21. Answer: $\boxed{1/4}$

Solution: The probability Yoon wins the n-th game is $1 - \frac{1}{n+2} = \frac{n}{n+2}$. Thus, the probability that he wins each of the first six games is $\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{7}{8}$. All but one of the top and bottom terms cancel, and we are left with $2/8 = 1/4$.

22. Answer: $\boxed{24/5}$

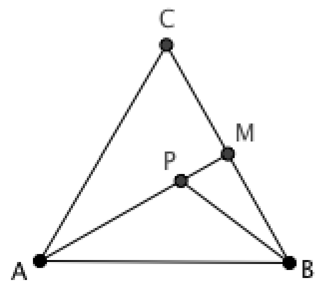


Solution: Let ω_1 and ω_2 be centered at O_1 and O_2, respectively. Note that $O_1A = 3$, since it is a radius of ω_1, and $O_2A = O_2B - BA = 2$. Thus, $O_1O_2 = 3 + 2 = 5$. Also, $O_1X = 3$ and $O_2X = 4$, so it follows that $O_1X^2 + O_2X^2 = O_1O_2^2$, that is, $\angle O_1XO_2 = 90°$. By symmetry, O_1YO_2 is also a right triangle, and the two triangles are congruent. Thus, $XY = 2XZ$, where Z is the intersection of XY and O_1O_2.
We can now compute the area of triangle O_1XO_2 in two ways. First, using base XO_1 and height XO_2, the area is $\frac{1}{2} \cdot 3 \cdot 4 = 6$. Now, it also has base $O_1O_2 = 5$ and height XZ, so $XZ = 2 \cdot \frac{6}{5} = \frac{12}{5}$, and so $XY = 2 \cdot \frac{12}{5} = \frac{24}{5}$.

23. Answer: $\boxed{720}$

Solution: We will put in the marbles one row at a time. Starting at the top, we have six possible positions for the first marble. Then, in the next row, we have six positions again, but one is taken, namely, the position directly underneath the first marble (since no two marbles may be in the same column). Thus, there are 5 possible places to put the next marble. In the third row, we have six positions, but two are taken, the ones underneath the first two marbles (which we know are already in different columns). Repeating this process, we find that our answer is $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

24. Answer: $\boxed{\sqrt{19}/8}$



Solution: We have that $MB = 1/2$ and since ABC is equilateral, $\angle AMB = 90°$. In particular, AMB is a 30-60-90 triangle, since $\angle ABM = 60°$. Thus, $AM = \sqrt{3}MB = \sqrt{3}/2$. Then, $PM = AM/4 = \sqrt{3}/8$. Applying the Pythagorean Theorem, we get $BP = \sqrt{PM^2 + MB^2} = \sqrt{3/64 + 1/4} = \sqrt{19/64} = \sqrt{19}/8$.

25-27. Answers: $\boxed{(26, 22, 8)}$ (each problem is still graded separately)
Solution: We'll start with problem 26, since it's the one with the most restrictions. If the two roots of the quadratic are m and n, we have $(x-m)(x-n) = x^2 - (m+n)x + mn = x^2 - Ax + 48$. Thus, $A = m + n$ and $48 = mn$. Since m and n are positive integers, we only have a few possibilities for m and n: they are either $\{1, 48\}$, $\{2, 24\}$, $\{3, 16\}$, $\{4, 12\}$, or $\{6, 8\}$ (changing the order of m and n doesn't affect their sum or positive difference). These give the corresponding values of $A = m + n$ of 49, 26, 19, 16, and 14

Taking a look at problem 25, we see that the answer has to be a positive integer with exactly four positive factors. This immediately eliminates 49, 19, and 16 from being values of A, since these have 3, 2, and 5 positive factors, respectively. Thus, either $A = 26$ or $A = 14$. Now, from here, we'll look at the corresponding values of B and C for these two possible values of A. The first few positive integers with exactly four factors, in order are: $6, 8, 10, 14, 15, 21, 22, 26$. Thus, for $A = 14$, we get $C = 4$, and $B = |6-8| = 2$, and for $A = 26$, we get $C = 8$, and $B = |2-24| = 22$.

To finish, we plug these values of B and C into problem 27 to see what works. If the first case gives the answer, we get that the smallest integer greater than $2/\pi$ is 4. However, this is clearly not true, because $\pi > 2$, so $2/\pi < 1$. In the other case, we want the smallest integer greater than $22/\pi$ to be 8. This looks plausible, because of the well-known approximation $22/7 \approx \pi$. Indeed, we have $7 < 22/\pi < 8$, because $\pi \approx 3.141\ldots$ and $22/7 \approx 3.142\ldots$, so $\pi < 22/7$, and $\pi > 3 > 22/8$, so $22/\pi < 8$. This gives the unique solution $(A, B, C) = (26, 22, 8)$.

28. Answer: $\boxed{1848}$
Solution: We will instead count the number of ways to make two knights attack each other, then subtract this from the total number of ways to place two knights on the board. Observe that if two knights attack each other, we can draw a 2×3 rectangle along the gridlines of the board so that the knights lie at two of its corners. Conversely, given a 2×3 rectangle on the board, we get two ways to make two knights attack each other, by placing them at pairs of opposite corners.

We now want to count the number of such rectangles. We only need to count the ones that are 2 units down and 3 across, because by symmetry, there are the same number of the other variety, 2 units across and 3 down. We'll multiply by 4 at the end, to count both of these types of rectangles, and account for the fact that we have 2 configurations for each rectangle. Note that on our board, we have $8 + 1 = 9$ lines in the horizontal and vertical directions. To get a rectangle with of dimensions 2 down and 3 across, we choose two vertical lines that are 3 units apart, and two horizontal lines that are 2 units apart, and their intersections will form the desired rectangle. There are 6 ways to pick the vertical lines and 7 to pick the horizontal lines (why?), so we have 42 rectangles, and $42 \cdot 4 = 168$ ways to make the knights attack each other.

To finish, we want to subtract this from $\binom{64}{2} = 32 \cdot 63 = 2016$, the total number of ways to choose two squares on the board. Thus, our answer is $2016 - 168 = 1848$.

29. Answer: $\boxed{0, 1, 3, 4, 5, 6}$
Solution: We can get the above number of intersections in the following ways:

0: all four lines parallel;
1. all four lines passing through the same point;
3: three lines parallel, and the fourth passing through all of them (as a transversal);
4: two pairs of parallel lines, forming a parallelogram;
5: two parallel lines with two transversals that intersect off the parallel lines;
6: four lines, no two parallel, no three passing through the same point.

To see why we can't have exactly two intersections, start by drawing two intersecting lines. If we draw a third line, it cannot intersect both lines in different places, or else we get 3 intersections, so it must either be parallel to one of the lines, or pass through the point of intersection of the first two. In the first case, we have two parallel lines and a transversal, but the fourth line will have to intersect one of the lines somewhere, giving more than 2 intersections. In the second case, the fourth line cannot pass through the point of intersection of the first three lines, or else we only have 1 intersection, but it can be parallel to at most one of the three lines, so we'll get at least 2 more intersections with the first three, for a total of at least 3 intersections. Either way, it's impossible to get exactly two, giving our answer.

30. Answer: $\boxed{1/69}$
Solution: We will first count the number of possible ways Dave could put back the books. To do this, we first start by systematically listing the positions (1 through 7, from left to right), in which he can put back the red books. Then, the number of spaces left between 1 and 4, inclusive, will be the number of ways to put back the yellow book. If he manages to get these four books in the right places, the probability of getting the green book into its correct place, out of three places left, will be 1/3; we thus multiply the probability of getting the correct arrangement for these first four books by 1/3 at the end.

1-3-5: 2 places for the green book

1-3-6: 2	2-4-6: 2
1-3-7: 2	2-4-7: 2
1-4-6: 2	2-5-7: 3
1-4-7: 2	3-5-7: 3
1-5-7: 3	

The probability of getting the red books and yellow book in the right place is thus $\frac{1}{2 \cdot 7 + 3 \cdot 3} = \frac{1}{23}$, and our answer is $\frac{1}{23} \cdot \frac{1}{3} = \frac{1}{69}$.

31. Answer: $\boxed{24}$
Solution: We let our sequence of letters be $a_1, a_2, \ldots, a_{11}$. We'll start by assigning arbitrary letters to a_1 and a_2, say $a_1 = X$ and $a_2 = Y$, where $\{X, Y, Z\} = \{L, M, T\}$ (note that a_1, a_2 cannot be the same, as they are consecutive multiples of 1). Next a_3 can either be X or Z.

In the first case, say $a_3 = X$. Then, $a_4 \neq a_2, a_3$, since we get consecutive multiples of 2 and 1, so $a_4 = Z$. $a_6 \neq a_5, a_4, a_3$, so a_5, a_4, a_3 cannot all be different, or else we'll have no letter for a_6. However, $a_5 \neq a_4$, and $a_4 \neq a_3$, so we need $a_5 = a_3 = X$. Then, $a_6 = Y$. Next, note that $a_8 \neq a_7, a_6, a_4$, and by similar logic to before, $a_7 = a_4 = Z$ and $a_8 = X$. $a_9 \neq a_8, a_6$, so $a_9 = Z$. $a_{10} \neq a_9, a_8, a_5$ so $a_{10} = Y$. Finally, $a_{11} \neq a_{10}$, so we can make $a_{11} = X, Z$. Thus, our sequence will look like $XYXZXYZXZY(X/Z)$, in which we can end with either X or Z.

Using the exact same logic from before, if $a_3 = Z$ instead, our sequence will look like $XYZXZYXZXY(X/Z)$. These two types of sequences are mutually exclusive, since the first has $a_1 = a_3$ and the second does not. Thus, each type of sequence breaks into two "sub-types", for choosing X or Z at the end. Then, there are 6 ways to determine how to assign $\{X, Y, Z\}$ to $\{L, M, T\}$, so our answer is $6 \cdot 2 \cdot 2 = 24$.

32. Answer: $\boxed{26}$
Solution: To make this solution a little more concise, we'll use summation notation for part of it. We'll write, for example, $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ to mean that we're adding up the terms of the sequence $\frac{n^3}{2^n}$ over all positive integers n (from 1 to "infinity"). Instead of computing $S_3 = \sum_{n=1}^{\infty} \frac{n^3}{2^n}$, we'll start by computing an easier sum, $S_0 = \sum_{n=1}^{\infty} \frac{n^0}{2^n}$.

S_0 is just $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, which is a geometric series with first term 1/2 and common ratio 1/2. Thus, we have $S_0 = \frac{1/2}{1/2} = 1$.

Now, we move on to $S_1 = \sum_{n=1}^{\infty} \frac{n^1}{2^n}$. We want to use self-similarity in S_1 to make S_1 look more like S_0, and to be able to compute S_1 using S_0. The trick is to multiply S_1 by 2, and compare it to S_1:

$$\begin{aligned} 2S_1 &= 1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots \\ S_1 &= 0 + \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots \end{aligned}$$

Now, everything lines up nicely to subtract S_1 from $2S_1$, to get just S_1. But this will give us $S_1 = 1 + S_0 = 2$.

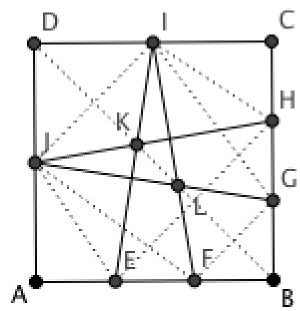
Now, to S_2; we'll try the same trick, but with summation notation to make things more concise. We have $S_2 = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$, and $2S_2 = \sum_{n=1}^{\infty} \frac{n^2}{2^{n-1}} = 1 + \sum_{n=1}^{\infty} \frac{(n+1)^2}{2^n}$ (here, we're just shifting the indices over by 1 to get the same denominators with matching indices between S_2 and $2S_2$). Now, we can subtract, and we see that

$$S_2 = 1 + \sum_{n=1}^{\infty} \frac{(n+1)^2 - n^2}{2^n} = 1 + \sum_{n=1}^{\infty} \frac{2n+1}{2^n} = 1 + 2S_1 + S_0 = 6.$$

Finally, in a similar way we get

$$S_3 = 1 + \sum_{n=1}^{\infty} \frac{(n+1)^3 - n^3}{2^n} = 1 + \sum_{n=1}^{\infty} \frac{3n^2+3n+1}{2^n} = 1 + 3S_2 + 3S_1 + S_0 = 26.$$

33. Answer: $\boxed{6\sqrt{2}/35}$



Solution: Consider quadrilateral $IHEJ$. Notice that IJ is parallel to EH, because EIJ and BHE are both 45-45-90 triangles, and that $JE = IH$, because triangles AEJ and CHI are congruent right triangles (notice that their legs have the same lengths). Thus, $IHEJ$ is an isosceles trapezoid, with K as the intersection of its diagonals. This means that K is on the perpendicular bisector of IJ and EH, which we see is just BD. In a similar way, L is also on BD.

Now, consider quadrilaterals $DIKJ$ and $BHKE$. We claim that they are similar, with the given vertex order. We have that triangles DIJ and BHE are similar, since they are both 45-45-90. Thus, $\frac{DI}{BH} = \frac{JD}{EB} = \frac{IJ}{HE}$. Also, by isosceles trapezoid $IHEJ$, we have $IKJ \sim HKE$ (we have equal angles from the isosceles condition and the parallel condition). Thus, $\frac{IJ}{HE} = \frac{KJ}{KE} = \frac{EB}{JD}$. But this means that when we compare the ratios of corresponding sides of $DIKJ$ and $BHKE$, they will always be the same, as we've found that they're all equal to IJ/EH. Thus, $DIKJ \sim BHKE$ and similarly (pun intended) $DILJ \sim BFLG$.

By the first similarity, it follows that $DK/KB = IJ/FG = DJ/GB = (1/2)/(2/3) = 3/4$. Thus, $DK/DB = 3/7$, and it follows that $DK = 3\sqrt{2}/7$. Now, $BL/DL = FG/IJ = (1/3)/(1/2) = 2/3$, so $BL/BD = 3/5$, and we get $BL = 3\sqrt{2}/5$. To finish, we have $KL = BD - DK - BL = \sqrt{2}(1 - 3/7 - 2/5) = 6\sqrt{2}/35$.

34. Answer: $\boxed{78735}$

Solution: The answer to this problem was programmed by computing $1 + \sum_{p \text{ prime}} \lfloor \log_p 10^6 \rfloor$. If you understand what logarithms do, try to figure out why this works.

35. Answer: $\boxed{130023}$

Solution: This question is equivalent to asking for the number of ways to *partially order* a set with six elements. This is similar to ordering the elements of the set in the usual way, except two elements don't necessarily have to have an order relation. For example, with the set $\{1, 2, 3, 4, 5, 6\}$, we usually think of $1 < 2 < 3 < 4 < 5 < 6$, but what do we do with the set $\{1, 2, Z, \text{LMT}, \text{AoPS}, \text{Shostakovich}\}$? Some pairs of elements just won't be comparable, and that's okay, we just leave them that way. The only requirement for a partially-ordered set is that the ordering is transitive, that is, if $x < y$ and $y < z$, we must have $x < z$ (look at how this translates into the way the original problem was worded).

Partially ordered sets are central to a field of mathematics known as Algebraic Combinatorics, which uses techniques of higher algebra to solve combinatorial problems. An enumeration of the number of partially ordered sets with a small number of elements can be found at
http://www.research.att.com/[tilde]njas/sequences/A001035.

36. Answer: $\boxed{???}$
Solution: On this problem, each of the five available scores were submitted at least twice by the sixteen teams, an event that happens with probability $\binom{10}{4}/\binom{20}{4} \approx 0.0433\ldots$. Oh well.